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Size effects in Kirchhoff flexible rods

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The Kirchhoff equations for flexible rods are extended to be able to describe size effects. Thus, the extended equations are applicable not only for macroscopic rods but also for ultrathin rods whose thickness is close to the material length. It is found that extensional size effects exist in flexible rods, namely, the external forces exerted on both ends of rods will increase as the rods are getting thinner if an identical deformation is remained.

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I. INTRODUCTION

In recent years, the Kirchhoff theory for flexible rods is widely used to model a variety of flexible curves, ribbons, cables, rods, and bars, even to model DNA (deoxyribonucleic acid), RNA (ribonucleic acid), and bacterial fibers, whether they have macroscopic or microscopic scale. However, any material has a characteristic length (material length), which depends on its microstructures. In perfect crystals, the material length is probably submicroscopic. In polycrystalline metals or granular materials, the material length may be considerably larger. For example, in the idealized case of a simple-cubic array of contiguous elastic spheres, the material length is about three quarters of the radius of a sphere. When rods get more slender and their thickness is close to the material length, their bending and twisting rigidity becomes larger than might be expected from the classical theory of elasticity. This phenomenon is referred to as the scale effects or size effects.

The material length of biomaterials has not thus far been reported in literatures. However, the microstructures of filamentlike biomaterials are observed in lots of experiments. Now it is known that DNA and RNA are polymers with a backbone consisting of a sugar-phosphate repeat unit, to each of which is attached one member of a small set of organic bases, generating the linear pattern of the genetic code. In the form of a double helix, DNA is about 2.0 nm in diameter. It might be expected that the material lengths of DNA and RNA are measured by the size of their backbone monomer. The smallest dimensions of DNA and RNA, their thickness, might approach the same scale of their material length.

In order to describe the size effects of material properties, the material length has to be introduced into the mechanical model as an intrinsic parameter. As a result, the couple stress theory (Toupin [1], Koiter [2], and Mindlin [3]) and the strain gradient theory (Fleck and Hutchinson [4] and Fleck *et al.* [5]) have been developed. Based on these theories, one can successfully describe the size effects of material properties not only in simple structures but also in composites, for example, in the particle reinforced aluminum (Kouzeli and Mortensen [6]), in nanoscale thin films (Haque and Saif [7]), and in the micropolar composite with fibers (Ma and Hu [8]). Additionally, the Cosserat theory is also used to predict the size effects in a variety of complex structures and composites, such as, the polycrystals and multiphase materials (Forest *et al.* [9]). For measuring the plasticity length scale, Stolken and Evans [10] proposed a microbend test method.

In the present paper, the material length as an intrinsic parameter is introduced into the Kirchhoff theory for elastic rods based on the couple stress theory. The extended theory can then be applied to the ultrathin rods whose thickness is close to their material length.

II. FLEXIBLE RODS

Consider a flexible slender rod with length of L. Suppose that each cross section of the rod remains plane and perpendicular to the rod's neutral axis during deformation. The rod's neutral axis coincides with the centroid of the rod's cross section and is a space curve.

Choose a set of principal axes of cross sections as the rod's local coordinate axes ($P-xyz$). The associated coordinate basis is ($P-\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{T}$), where the base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are along the principal axes of cross sections and $\boldsymbol{T}$ is the tangent vector along the rod's neutral axis.

The rotation rate of cross sections with respect to the fixed reference frame from the point of view of an observer moving along the neutral axis at unit speed is denoted in terms of

$$\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2 + \omega_3\boldsymbol{T}, \tag{1}$$

which is the absolute rotation rate of cross sections and can be determined by use of the Darboux vector.

The evolution of the coordinate basis ($P-\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{T}$), the principal axes of cross sections, can be described by

$$\frac{d\boldsymbol{e}_1}{ds} = \boldsymbol{\omega} \times \boldsymbol{e}_1, \tag{2a}$$

$$\frac{d\boldsymbol{e}_2}{ds} = \boldsymbol{\omega} \times \boldsymbol{e}_2, \tag{2b}$$

$$\frac{d\boldsymbol{T}}{ds} = \boldsymbol{\omega} \times \boldsymbol{T}. \tag{2c}$$

III. EQUILIBRIUM EQUATIONS

Besides the assumption of "plane sections remain plane," we suppose that rods are homogeneous and isotropic. There

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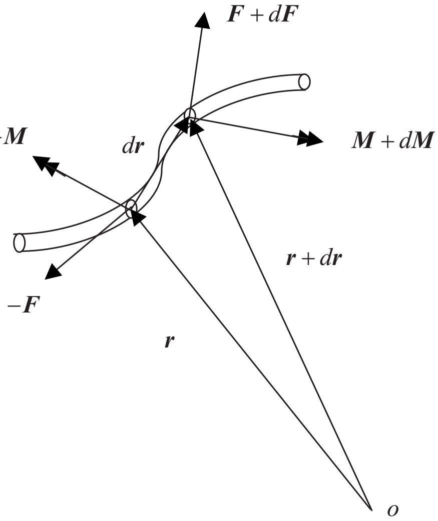

FIG. 1. Internal forces and moments.

exists a linear relationship between stresses and strains in rods. Volume forces and the contact forces with other rods are neglected. Rods do not have initial curvature and twist.

Consider an infinitesimal element of rods that is of length ds shown in Fig. 1. The internal forces $\boldsymbol{F}$ and moments $\boldsymbol{M}$ applied on the both ends satisfy the equilibrium equations

$$\frac{d\boldsymbol{F}}{ds}=0, \tag{3a}$$

$$\frac{d\boldsymbol{M}}{ds}+\boldsymbol{T}\times\boldsymbol{F}=0. \tag{3b}$$

After a transformation to the principal-axis coordinates ($P-\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{T}$), the above equations become

$$\frac{d\boldsymbol{F}}{ds}+\boldsymbol{\omega}\times\boldsymbol{F}=0, \tag{4}$$

$$\frac{d\boldsymbol{M}}{ds}+\boldsymbol{\omega}\times\boldsymbol{M}+\boldsymbol{T}\times\boldsymbol{F}=0. \tag{5}$$

Equation (3a) implies that the internal forces $\boldsymbol{F}$ are a constant vector in the fixed reference frame. Let the fixed coordinate axis ζ be parallel to the direction of $\boldsymbol{F}$. Denoting the direction cosines of the ζ axis with respect to each axis of $(P-xyz)$ in terms of γ_1, γ_2, and γ, respectively, we have

$$F_1=F\gamma_1,\quad F_2=F\gamma_2,\quad F_3=F\gamma, \tag{6}$$

where F_1, F_2, and F_3 are the components of $\boldsymbol{F}$ in the principal-axis coordinates and $F=|\boldsymbol{F}|$. Substituting Eqs. (1) and (6) into Eq. (4) gives the equation of equilibrium of forces in the component form

$$\frac{d\gamma_1}{ds}+\omega_2\gamma-\omega_3\gamma_2=0, \tag{7a}$$

$$\frac{d\gamma_2}{ds}+\omega_3\gamma_1-\omega_1\gamma=0, \tag{7b}$$

$$\frac{d\gamma}{ds}+\omega_1\gamma_2-\omega_2\gamma_1=0. \tag{7c}$$

Similarly, the equation of equilibrium of moments (5) can be written in the component form

$$\frac{dM_1}{ds}+\omega_2M_3-\omega_3M_2-F_2=0, \tag{8a}$$

$$\frac{dM_2}{ds}+\omega_3M_1-\omega_1M_3+F_1=0, \tag{8b}$$

$$\frac{dM_3}{ds}+\omega_1M_2-\omega_2M_1=0, \tag{8c}$$

where M_1, M_2, and M_3 are the components of $\boldsymbol{M}$ in the principal-axis coordinates.

Equations (7) and (8) together are six equations for the nine unknowns. We may eliminate M_1, M_2, and M_3 in Eq. (8) by using the linear relationship between internal moments $\boldsymbol{M}$ and curvatures $\boldsymbol{\omega}$, which is

$$M_1=a\omega_1,\quad M_2=b\omega_2,\quad M_3=c\omega_3, \tag{9}$$

where a and b are the bending rigidity about the principal axes x and y of cross sections, respectively, and c is the twisting rigidity about the z axis, i.e., the rod's axis. Substituting Eq. (9) into Eq. (8) yields

$$a\frac{d\omega_1}{ds}+(c-b)\omega_2\omega_3-F\gamma_2=0, \tag{10a}$$

$$b\frac{d\omega_2}{ds}+(a-c)\omega_3\omega_1+F\gamma_1=0, \tag{10b}$$

$$c\frac{d\omega_3}{ds}+(b-a)\omega_1\omega_2=0. \tag{10c}$$

In order to determine the rigidities a, b, and c, the pure bending of a prismatic beam and the torsion of a cylindrical rod has to be discussed.

IV. BENDING RIGIDITY AND SIZE EFFECTS

Consider the pure bending of a prismatic beam of an isotropic Hookean material. Suppose the beam at its ends is subjected to two equal and opposite couples $\boldsymbol{M}$ acting in its principal plane of bending. Let the origin of the coordinates be taken at the centroid of a cross section and let the x,y plane be the principal plane. The usual theory of bending assumes that the stress components are

$$\sigma_{zz}=\frac{Ey}{R},\quad \sigma_{xx}=\sigma_{yy}=\tau_{xy}=\tau_{yz}=\tau_{zx}=0, \tag{11}$$

in which E is Young's modulus and R is the radius of curvature of the beam after bending. The bending moment about x axis due to the normal stress σ_{zz} in Fig. 2 is therefore

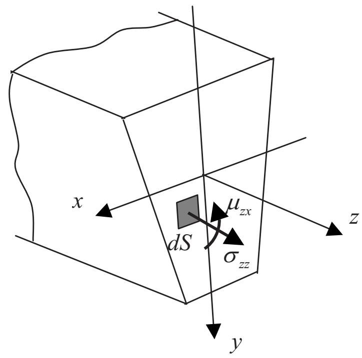

FIG. 2. Normal stress and couple stress acting on an element dS.

$$M^{\sigma} = \int_S y\sigma_{zz} dS = \frac{EI_x}{R}, \tag{12}$$

where I_x is the moment of inertia of the beam's cross-sectional area about the x axis and S is the beam's cross-sectional area.

Now let us suppose, according to the couple stress theory developed by Mindlin [3], that not only the normal stress σ_{zz} but also a couple stress μ_{zx} acts on an element dS of the beam's cross section as shown in Fig. 2. Moreover, the couple stress is proportional to the curvature

$$\begin{cases} \mu_{zx} = \dfrac{B_1}{R} \\ \mu_{xx} = \mu_{yy} = \mu_{zz} = \mu_{xy} = \mu_{yx} = \mu_{yz} = \mu_{zy} = \mu_{xz} = 0, \end{cases} \tag{13}$$

where B_1 is a modulus of curvature. Note that the distribution of the couple stress μ_{zx} on the cross section is uniform because the radius of curvature R at an arbitrary point of the beam is identical.

Obviously, integrating the couple stress μ_{zx} over the cross section also yields a bending moment

$$M^{\mu} = \int_S \mu_{zx} dS = \mu_{zx} S. \tag{14}$$

By substituting Eq. (13) into Eq. (14), we obtain

$$M^{\mu} = \frac{B_1}{R} S. \tag{15}$$

The total bending moments consist of both M^{σ} and M^{μ}. They are equal to the external couples M acting on the ends of the beam, so that

$$M = M^{\sigma} + M^{\mu} = \frac{(EI_x + B_1 S)}{R}. \tag{16}$$

Similar to the way indicated by Mindlin [3], a material length l can be defined in terms of the ratio

$$\frac{B_1}{E} = l^2. \tag{17}$$

Then, Eq. (16) becomes

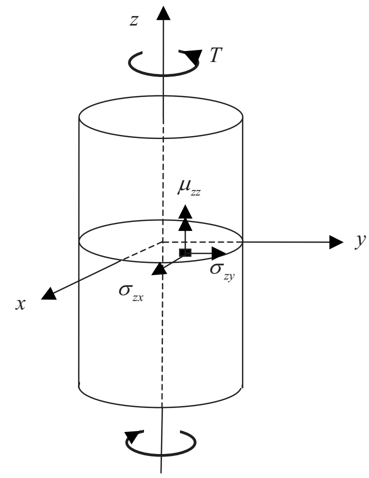

FIG. 3. Torsion of a circular rod.

$$\frac{M}{E[I_x + l^2 S]} = \frac{1}{R}. \tag{18}$$

It can be seen that the bending rigidity of the beam becomes $E[I_x + l^2 S]$ instead of EI_x in the Bernoulli beam theory if the couple stresses are taken into account. The new term $l^2 S$ in the bending rigidity expresses size effects.

When the cross section of beams is circular, the bending rigidity becomes

$$E[I_x + l^2 S] = EI_x\left[1 + 4\left(\frac{l}{\delta}\right)^2\right], \tag{19}$$

where δ is the rod's diameter.

Thus, the bending rigidities of the flexible slender rods a and b in Eq. (10) can finally be written in the form

$$a = A\left[1 + 4\left(\frac{l}{\delta}\right)^2\right] \quad \text{and} \quad b = B\left[1 + 4\left(\frac{l}{\delta}\right)^2\right], \tag{20}$$

in which

$$A = EI_x, \quad B = EI_y. \tag{21}$$

V. TWISTING RIGIDITY AND SIZE EFFECTS

Let us consider the torsion of a cylindrical rod of circular cross section. See Fig. 3, which shows the notations and the ordinate axes to be used. The rod is acted on its ends by a torque T.

The nonvanishing components of stresses in cross sections are the shears σ_{zx} and σ_{zy}, from which a symmetric part can be obtained as

$$\sigma_{zx}^s = \frac{1}{2}(\sigma_{zx} + \sigma_{xz}), \sigma_{zy}^s = \frac{1}{2}(\sigma_{zy} + \sigma_{yz}). \tag{22}$$

They obey the usual stress-strain relation

$$\sigma_{zx}^s = -G\theta y, \quad \sigma_{zy}^s = G\theta x, \tag{23}$$

where G is shear modulus and θ is the twist per unit axial length. In addition, based on the couple stress theory, there is a couple stress μ_{zz} in cross sections.

The linear relationship between μ_{zz} and θ is written as

$$\mu_{zz} = B_2\theta, \tag{24}$$

where the proportional constant B_2 and the modulus of curvature B_1 are related by

$$B_2 = \frac{B_1}{1+\nu}. \tag{25}$$

The resultant moment about the z axis is

$$\int\int (x\sigma^s_{zy} - y\sigma^s_{zx})dxdy + \int\int \mu_{zz}dxdy. \tag{26}$$

Substituting Eqs. (23)–(25) into Eq. (26) and letting the resultant moment be equal to the torque T yield

$$T = GI_z\theta + \frac{B_1 S\theta}{1+\nu}, \tag{27}$$

where I_z is the polar moment of inertia of the cylinder cross-sectional area.

By introducing the material length l from Eq. (17), Eq. (27) is rewritten as

$$T = G(I_z + 2l^2S)\theta. \tag{28}$$

For the circular cross sections, we have

$$T = GI_z\left[1+4\left(\frac{l}{\delta}\right)^2\right]\theta. \tag{29}$$

Obviously, the twisting rigidity becomes $GI_z[1+4(\frac{l}{\delta})^2]$ in the extended theory instead of GI_z as in classical elasticity when the size effects are taken into account. Thus, the twisting rigidity of the flexible slender rod c in Eq. (10) can finally be written in the form

$$c = C\left[1+4\left(\frac{l}{\delta}\right)^2\right], \tag{30}$$

in which

$$C = GI_z. \tag{31}$$

Substituting Eqs. (20) and (30) into Eq. (10) gives

$$A\frac{d\omega_1}{ds} + (C-B)\omega_2\omega_3 - \frac{1}{\left[1+4\left(\frac{l}{\delta}\right)^2\right]}F\gamma_2 = 0, \tag{32a}$$

$$B\frac{d\omega_2}{ds} + (A-C)\omega_3\omega_1 + \frac{1}{\left[1+4\left(\frac{l}{\delta}\right)^2\right]}F\gamma_1 = 0, \tag{32b}$$

$$C\frac{d\omega_3}{ds} + (B-A)\omega_1\omega_2 = 0. \tag{32c}$$

Equations (7) and (32) are the final equations.

It can be seen that an additional factor $[1+4(\frac{l}{\delta})^2]$ arises in the above equations in comparison to the Kirchhoff equations [11]. Obviously, the factor expresses the size effects. When the rod's diameter δ is greatly larger than the material length l, the size effect term $4(\frac{l}{\delta})^2$ is negligible and then the above equations reduce to the Kirchhoff equations. Contrarily, when the rod's diameter δ is close to the material length l, the size effect term $4(\frac{l}{\delta})^2$ has to be taken into account and even becomes a leading term in the factor $[1+4(\frac{l}{\delta})^2]$. It is well known that the material length of most metals is of the order of 1 micron. This indicates that the extended equation in the present paper is applicable for the metallic rods of micron scale in thickness. Moreover, the extended equations in the present paper seem to be applicable for modeling DNA, RNA, and bacterial fibers instead of the Kirchhoff equations.

VI. SIZE EFFECTS IN EXTENSIONAL DEFORMATION

It can be verified that the extended Eqs. (7) and (32) can be obtained in the simple way that the magnitude of the internal forces F in the Kirchhoff equations are divided by the factor $[1+4(\frac{l}{\delta})^2]$. This means that the external forces exerted on both ends of rods have to be magnified by the factor $[1+4(\frac{l}{\delta})^2]$ to remain an identical deformation of rods because the bending and twisting rigidities of the rods increase by the same factor as shown in Eqs. (20) and (30). Obviously, the factor expresses the size effects. Thus, one would feel that the extensional rigidity of the ultrathin rods seems to be larger than might be expected from the classical elasticity when the rods are getting thinner and the diameter of rods is close to the material length.

VII. CONCLUSIONS

(1) The Kirchhoff equations for flexible rods are extended to Eqs. (7) and (32) in the present paper, which are applicable not only for macroscopic flexible rods but also for so ultrathin flexible rods whose thicknesses are close to the material length.

(2) The extended equations can be easily obtained by dividing the magnitude of internal forces F in the Kirchhoff equations by the factor $[1+4(\frac{l}{\delta})^2]$, for the rods of circular cross sections, in which the term $4(\frac{l}{\delta})^2$ measures the size effects

(3) The extensional rigidity of ultrathin rods would be larger than might be expected from the classical elasticity. This is because the increase of the bending and twisting rigidities requires that the external force exerted on both ends of the rod be increased by an equal factor to countervail the decrease of the internal forces so that an identical deformation of the rods is maintained. This is the size effects in extensional deformation of rods.

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